G T 1

2020/11/10

Jiao

P1

Régularité local & short-in-time smooth.

2020/11/10

I : Introduction :

- N-S : $\begin{cases} \partial_t u - \nu \Delta u + u\cdot\nabla u + \nabla P = 0 \\ \underline{\operatorname{div} u = 0} . \quad u(x,0) = u_0 . \end{cases}$ dans $Q_+ = \mathbb{R}^3 \times]0,+\infty[$.

Scaling : $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$

$P_\lambda(x,t) = \lambda^2 P(\lambda x, \lambda^2 t)$.

exemples : $\mathbb{R}^d$

$L^\infty_t H^{\frac{d}{2}-1} \cap L^2_t \dot{H}^{\frac{d}{2}}$

$d=2$: $L^\infty_t L^2_x \cap L^2_t \dot{H}^1$

$d=3$: $L^\infty_t H^{\frac12}_x \cap L^2_t H^{\frac32}$

- la solution faible de Leray (34').

① plus de regularité : $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$

② Inégalité d'énergie : $\int |u|^2 + \int |\nabla u|^2 \underset{\Delta}{\leq} \int |u_0|^2$.

- la sol forte : u est une sol forte dans $Q_T = \mathbb{R}^3 \times]0,T[$.

$\Longleftrightarrow \quad \sup_{0<t<T} \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 dx < +\infty$

si u et u' sont deux sol faible de Leray-Hopf,

u est une sol forte. $\Rightarrow$ $u = u'$ dan Q_T

- Régularité & Unicité :

régularité $\Rightarrow$ unicité

- Conditions de Serrin (62'), Ladyzhenskaya (68'). Prodi (59')

$u \in L^\infty_t L^2_x (Q_1)$ $\quad w = \nabla\times u \in L^2_t L^2_x (Q_1)$. $\quad L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$

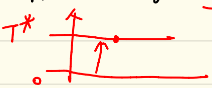
Si de plus on a :

$u \in L^r_t L^p_x (Q_1)$, $\quad \underbrace{\frac{2}{r} + \frac{3}{p} \leq 1 .}_{\text{subcritical \& critical}}$ $\quad p \in (3,\infty]$

Alors. $u \in L^\infty_{t,x} (Q_{\frac12})$

- Endpoint $L^\infty_t L^3_x$, Escauriaza. Seregin. Sverak (03').

Supposons que T^* est le 1er temps blow-up. Alors, $\limsup_{t\to T^*} \|u(\cdot,t)\|_{L^3} = \infty$



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- Endpoint $L^\infty_t L^3_x$, Escauriaza. Seregin. Sverak (03').

Supposons que T^* est le 1er temps blow-up. Alors, $\limsup_{t\to T^*} \|u(\cdot,t)\|_{L^3} = \infty$

$\Updownarrow$

Si $\limsup_{t\to T^*} \|u(\cdot,t)\|_{L^3} < \infty$ alors u n'est blow-up à T^*.

✓

blow-up

$$\boxed{\begin{array}{l} 0 < g(z_0) = \limsup_{R\to 0} \frac{1}{R} \sup \int_{B(z_0,R)} |u|^2 \, dz \quad R \\ \text{Type I : si } g(z_0) < \infty \\ \text{Type II : si } g(z_0) = \infty. \end{array}}$$

$\to \cdot L^\infty L^3. \quad L^2$

blow-up à T^*. $\|u\|_{L^\infty_t L^3_x(\mathbb{R}^3 \times (0,T^*))} \leq C$. (Tao. 2019)

blow-up à T^*. $\|u\|_{L^\infty_t L^{3,\infty}_x(\mathbb{R}^3 \times (0,T^*))} \leq C$. (Christophe Barker 2020)

Seregin 2012 : T^* est 1er temp blow-up.

$\|u(\cdot,t)\|_{L^3} \longrightarrow \infty, \quad t \to T^*.$

$$\lim_{t\to T^*} \|u(\cdot,t)\|_{L^3} = \infty.$$

Seregin 2020 : Axis-Sym + singular point $\Rightarrow$ non type I blow-up

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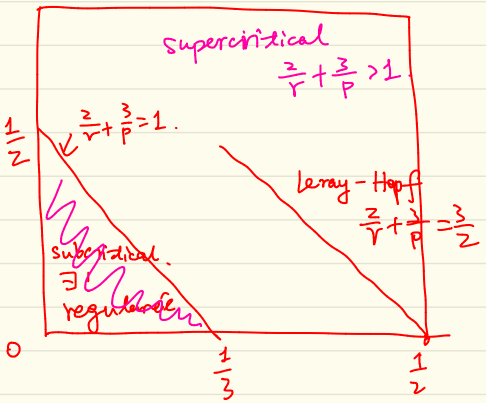
Subcritical: $p \in (3, \infty]$.

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \gtrsim (T^* - t)^{-\frac{3}{2}\left(\frac{1}{3}-\frac{1}{p}\right)}$$

avec $t \in (0, T^*)$.

$p = 3$. Seregin (2014')

No! une fonction



$\rightarrow$ Inégalité d'énergie.

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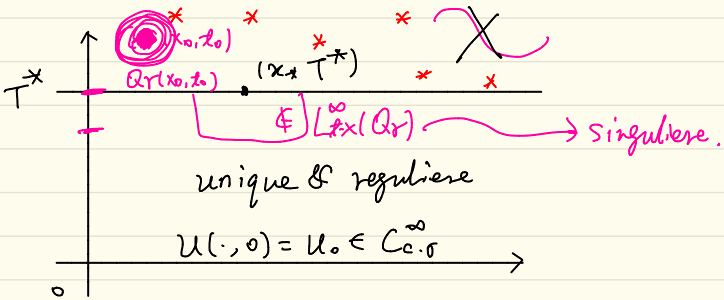
II: Blow-up ? Non unicité des sol faible avec energie finie.



Thm CKN : si l'ensemble de point singulière n'est pas $\emptyset$.
alors, il ne peut pas être une courbe.

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À faire plus tard.

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Blow-up. Sol auto-similaire

Jia & Sverak.

Forward. Autosimilaire :

$$\lambda u(\lambda^2 t, \lambda x) = u(t.x). \qquad \lambda > 0.$$

$$\Rightarrow \quad u(t.x) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right) \quad \text{avec} \quad U(x) = u(1,x)$$

Backward Auto-similaire :

$$\lambda u(\lambda^2 t, \lambda x) = u(t.x) \qquad \lambda > 0$$

$$\Rightarrow \quad u(t.x) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right). \quad \text{avec} \quad \underline{U(x) = u(-1, x)}$$

$$= \frac{1}{\sqrt{t}} u\left(-1, \frac{x}{\sqrt{|t|}}\right)$$

Q: negative time for NS

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Tsai 2019.

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III : Régularité partielle. (Caffarelli, Kohn, Nirenberg, 82', Lin 98')

Jia & Sverak

B.P 2019

Hypotheses for the Caffarelli-Kohn-Nirenberg regularity criterion

Definition 13.4
We call ($\mathcal{H}_{CKN}$) the following set of hypotheses:

1. *$\vec{u}$, p and $\vec{f}$ are defined on a domain $\Omega \subset \mathbb{R} \times \mathbb{R}^3$*
2. *on Ω, $\vec{u}$ belongs to $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$:*

$$\sup_{t\in\mathbb{R}} \int_{(t,x)\in\Omega} |\vec{u}(t,x)|^2\, dx < +\infty \text{ and } \iint_\Omega |\vec{\nabla} \otimes \vec{u}|^2\, dt\, dx < +\infty$$

3. *for some $q_0 > 1$, p belongs to $L_t^{q_0} L_x^1(\Omega)$:*

$$\int_{\mathbb{R}} \Big(\int_{(t,x)\in\Omega} |p(t,x)|\, dx\Big)^{q_0}\, dt < +\infty$$

4. *on Ω, $\vec{f}$ is a divergence free vector field in $L_{t,x}^{10/7}(\Omega)$:*

$$\operatorname{div} \vec{f} = 0 \text{ and } \iint_\Omega |\vec{f}(t,x)|^{10/7}\, dt\, dx < +\infty$$

5. *$\vec{u}$ is a solution of the Navier–Stokes equations on Ω:* $\operatorname{div} \vec{u} = 0$ *and*

$$\partial_t \vec{u} = \nu\Delta\vec{u} - \vec{u}.\vec{\nabla}\vec{u} + \vec{f} - \vec{\nabla} p \text{ in } \mathcal{D}'(\Omega) \tag{13.16}$$

$$\mu = -\partial_t|\vec{u}|^2 + \nu\Delta|\vec{u}|^2 - 2\nu|\vec{\nabla}\otimes\vec{u}|^2 + 2\vec{u}.\vec{f} - \operatorname{div}((|\vec{u}|^2 + 2p)\vec{u}) \tag{13.22}$$

K

is well defined on Ω.

Suitable solutions

Definition 13.5
The solution $\vec{u}$ is suitable *if the distribution μ is a non-negative locally finite measure on Ω.*

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Caffarelli-Kohn-Nirenberg regularity criterion

Theorem 13.8
Let Ω be a domain of $\mathbb{R}\times\mathbb{R}^3$. Let $(\vec{u},p)$ a weak solution on Ω of the Navier–Stokes equations

$$\partial_t\vec{u} = \nu\Delta\vec{u} - \vec{u}.\vec{\nabla}\vec{u} + \vec{f} - \vec{\nabla}p, \quad \operatorname{div}\vec{u} = 0.$$

Assume that

- $(\vec{u},p,\vec{f})$ *satisfies the conditions* $(\mathcal{H}_{CKN})$: $\vec{u}\in L^\infty L^2\cap L^2\dot{H}^1(\Omega)$, $p\in L^{q_0}L^1(\Omega)$ $(q_0>1)$, $\operatorname{div}\vec{f}=0$ *and* $\vec{f}\in L^{10/7}L^{10/7}(\Omega)$
- $\vec{u}$ *is suitable*
- $1_\Omega(t,x)\vec{f}\in\mathcal{M}_2^{10/7,\tau_0}$ *for some* $\tau_0>5/2$.

There exists a positive constant ϵ^* which depends only on ν and τ_0 such that, if for some $(t_0,x_0)\in\Omega$, we have

$$\limsup_{r\to0}\frac{1}{r}\iint_{(t_0-r^2,t_0+r^2)\times B(x_0,r)}|\vec{\nabla}\otimes\vec{u}|^2\,ds\,dx<\epsilon^*$$

then $\vec{u}$ is Hölderian (with respect to the quasi-norm $\delta(t,x)=|t|^{1/2}+|x|$) in a neighborhood of (t_0,x_0).

CKN: $\int_{Q_1}|u|^3+|P|^{3/2} =: \varepsilon < \varepsilon_{uni} \Rightarrow \|u\|_{L^\infty(Q_{\frac12})} \le C_{uni}\,\varepsilon^{1/3}$ qualitatif.

Quantitative reg: under small critical control.

Assume $\|u\|_{L^5(\mathbb{R}^3\times(-1,0))} \ll 1$. For all $x\in\mathbb{R}^3$.

$$\|u\|_{L^\infty(Q_{\frac12}(x,0))} \le C_{univ}\Big(\int_{Q_1(x,0)}|u|^3+|P|^{\frac32}\Big)^{1/3} \lesssim \|u\|_{L^5(\mathbb{R}^3\times(-1,0))}$$

$$\Rightarrow \|u\|_{L^\infty(\mathbb{R}^3\times(-\frac12,0))} \lesssim G(\|u\|_{L^5(\mathbb{R}^3\times(-1,0))}),\quad G(x)=x.$$

linear

P7: L^5 norm. $\quad Q_1(x,0) = B_1(0) \times (-1,0)$

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CKN: $\int_{Q_1} |u|^3 + |P|^{3/2} =: \varepsilon < \varepsilon_{uni} \Rightarrow \|u\|_{L^\infty(Q_{\frac{1}{2}})} \le C_{uni}\, \varepsilon^{1/3}$ qualitatif.

Quantitative reg: under small critical control.

Assume $\|u\|_{L^5(\mathbb{R}^3 \times (-1,0))} << 1$. For all $x \in \mathbb{R}^3$.

$$\|u\|_{L^\infty(Q_{\frac{1}{2}}(x,0))} \le C_{univ}\left(\int_{Q_1(x,0)} |u|^3 + |P|^{\frac{3}{2}}\right)^{1/3} \lesssim \|u\|_{L^5(\mathbb{R}^3 \times (-1,0))}$$

$$\Rightarrow \|u\|_{L^\infty(\mathbb{R}^3 \times (-\frac{1}{2},0))} \lesssim G(\|u\|_{L^5(\mathbb{R}^3 \times (-1,0))}), \quad G(x) = X.$$

linear

Tao.

1):

$$\|u\|_{L^\infty(\mathbb{R}^3 \times (-\frac{1}{2},0))} \lesssim \exp\exp\exp\left(\|u\|_{L^\infty(-1,0,L^3(\mathbb{R}^3))}\right)$$

$$:= G\left(\|u\|_{L^\infty_t L^3_x(-1,0,L^3(\mathbb{R}^3))}\right)$$

avec $G(X) = \exp\exp\exp(X)$

2): blow-up:

Supposons T^* 1er temps blow-up. Alors. $\lim_{t \to T^*} \frac{\|u\|_{L^3(\mathbb{R}^3)}}{\left(\log\log\log\left(\frac{c}{T^*-t}\right)\right)^c} = \infty$

Sub critical: $p \in (3, \infty]$.

simple: $\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \gtrsim \frac{1}{(T^*-t)^{\frac{3}{2}(\frac{1}{3}-\frac{1}{p})}}$.

P8 :

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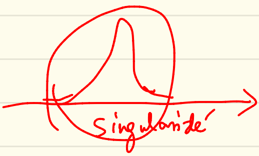
Tao 2019	B.P 2020 plus precise
global. $\mathbb{R}^3$.	local. Q_T.
$L^\infty L^3$	$L^\infty L^{3,\infty}$.
Fourier fréquency.	Jia & Sverak local-in-space short time smoothing
propagation backward.	propagation backward.
	continuation unique. quantitative.



Jia & S. 2014. ↳ résultat qualitatif.

B. P 2020. ↳ résultat quantitative.

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la method :

$u = a + V$, a est une sol mild (avec $u_0|_{B_0(1)} \in L^{5+\varepsilon}$)

V est une perturbation.

étape 1 : Estimations d'énergie locale pour V.

étape 2 : ε-régularité avec un subcritical drift.

(via Thy CKN : iteration.

ou Thy Lin : compacité).

Eq-perturbation :

$$\partial_t V - \Delta V + \underline{a\cdot\nabla V + \operatorname{div}(a\otimes V) + V\cdot\nabla V + \nabla q} = 0$$

① $\partial_t u - \Delta u + u\cdot\nabla u + \nabla P = 0$

② $\partial_t a - \Delta a + a\cdot\nabla a + \nabla\tilde{p} = 0$.

① − ② ⟹ $\partial_t(\underbrace{u-a}_{V}) - \Delta(\underbrace{u-a}_{V}) + \underline{u\cdot\nabla u - a\cdot\nabla a + \nabla p - \nabla\tilde{p}} = 0$

B-P 2019 : ils ont traité $L^5_{t,x}$ en utilisant un method d'itération.